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Stability of Nonstationary States of Classical, Many-Body Dynamical Systems

G. Grinstein[1]

We summarize recent arguments which show that for a broad class of classical, many-body dynamical model systems with short-range interactions (such as coupled maps, cellular automata, or partial differential equations), collectively chaotic states—nonstationary states wherein some Fourier amplitude varies chaotically in time—cannot occur generically. While chaos occurs ubiquitously on a *local* level in such systems, the macroscopic state of the system typically remains periodic or stationary. This implies that the dimension D of chaotic ("strange") attractors must diverge with the linear size L of the system like $D \sim (L/\xi)^d$ in d space dimensions, where ξ $(<\infty)$ is the spatial coherence length. We also summarize recent work which demonstrates that in spatially isotropic systems that have short-range interactions and evolve (like coupled maps) in discrete time, periodic states are never stable under generic conditions. In spatially anisotropic systems, however, short-range interactions that exploit the anisotropy and so allow for the stabilization of periodic states do exist.

KEY WORDS: Chaos; periodicity; attractor dimension; dynamical systems.

1. INTRODUCTION

A large class of classical many-body dynamical systems (e.g., Rayleigh–Benard convection, Taylor–Couette flow, cellular automata) evolve in time according to equations that possess either a discrete or continuous time-translation invariance. It is generally taken for granted that such systems can, even in the presence of noise, exhibit stable, nonstationary states—stable states wherein this invariance is spontaneously broken—so that some spatial Fourier amplitude is nonstationary (i.e., continues to vary with time even in the asymptotically long-time limit). In the

[1] IBM Research Division, T. J. Watson Research Center, Yorktown Heights, New York 10598.

surface wave experiments of Ciliberto and Gollub,[1] e.g., the bottom of a vessel containing fluid is oscillated in time at fixed frequency. Sufficiently vigorous oscillations make the free surface of the fluid unstable with respect to the formation of surface standing waves of appropriate wavelength. The amplitudes of these waves are observed to vary either chaotically in time or periodically, with periods different from the driving frequency. Similar phenomena are observed in many other experiments.[2]

From the theoretical point of view, evidence for the existence of such symmetry-breaking nonstationary states comes almost entirely from mean-field theories—theories that replace the infinite number of interacting modes of a many-body dynamical system in the thermodynamic limit by a finite (typically small) number.[3] The behavior can then be analyzed, either analytically or numerically. (The Lorenz equations, a finite-mode approximant to the problem of convection, provide perhaps the best-known example of this stategy.[6]) Nonstationary behavior of the amplitudes of the modes retained is a common result of the procedure, and is typically taken to imply the existence of stable nonstationary states in the original untruncated model.[4]

In this paper we consider some of the effects of the fluctuations omitted in mean-field treatments. We show how, under a broad range of conditions, these fluctuations actually destabilize the nonstationary states predicted by mean field theory. We first summarize (Section 2) recent arguments[8] which imply that, for a large class of models, collectively chaotic states—nonstationary states wherein some Fourier amplitude varies chaotically in time—simply cannot occur generically in systems with short-range interactions. (This is not to say that chaos cannot occur in many-body dynamical systems; such an assertion would be absurd. The claim is only that, while chaos can and does occur ubiquitously on a local level, the *macroscopic* state of typical many-body systems that are locally chaotic is periodic or stationary.) Then (Section 3) we point out a simple consequence of this result, which bears directly on the interpretation of experiments designed to measure the dimension of chaotic ("strange") attractors: The absence of collective chaos implies that attractor dimensions D must diverge with the linear size L of the sample like $D \sim (L/\xi)^d$, where $d = 1, 2$, or 3 is the system's spatial dimension and ξ $(<\infty)$ its

[2] See, e.g., Giglio *et al.*[2] for measurements on Rayleigh–Benard convection and Brandstater and Swinney[3] for measurements on Taylor vortex flow.

[3] For example, Ciliberto and Gollub[4] and Meron and Procaccia[5] discuss finite mode approximants to the surface wave problem of ref. 1.

[4] Curry *et al.*[7] study the effect of increasing the number of modes n in finite-mode approximants to Rayleigh–Benard convection. In two dimensions they find that the chaotic states that occur for small n disappear as n increases.

spatial coherence length. Finally (Section 4), we summarize recent work[9] which demonstrates that unless special care is taken in constructing the interactions of model many-body systems, even *periodic* states cannot be stabilized under generic conditions: In spatially isotropic systems that have short-range interactions and evolve (like coupled map lattices or cellular automata), in discrete time, periodic states with periods greater than two are never stable generically. In spatially anisotropic systems, however, short-range interactions that exploit the anisotropy and so allow for the stabilization of periodic states do exist. The generalization of these arguments about temporal periodicity to continuous-time systems is more complicated, and will be reported elsewhere.

2. COLLECTIVELY CHAOTIC STATES

Let us briefly recall the argument of ref. 8 for the absence of collective chaos in a broad class of many-body dynamical systems under generic conditions. Imagine a general many-body dynamical system described by scalar variables $x(\mathbf{r}, t)$ which take on real values roughly confined to the interval $0 \leqslant x(\mathbf{r}, t) \leqslant \Delta x$, say.[5] The system evolves according to some dynamical rule (e.g., partial differential equation), which for the moment is assumed noiseless (i.e., deterministic) and local (i.e., has no long-range interactions). Imagine that a particular variable $x(\mathbf{r}_1, t)$ is perturbed by an infinitesimal amount δx_0 at some time $t = 0$, say, and then allowed to evolve without further external interference. If the system is in the chaotic regime, then the value of $x(\mathbf{r}_1, t)$ at large subsequent times t differs from the value it would have had in the absence of the perturbation by an amount δx_t given by $\delta x_t \sim \delta x_0 \exp(\lambda t)$, where λ is some positive, Liapunov exponent which characterizes the sensitivity to perturbations or initial conditions of the chaotic state.[12] When t is so large that δx_t is of the order of the maximum range Δx of allowed values of $x(\mathbf{r}, t)$, i.e., when

$$t \sim t^* \equiv (1/\lambda) \ln(\Delta x/\delta x_0) \tag{1}$$

$x(\mathbf{r}_1, t)$ has been completely "dephased" by the perturbation, in that it has no memory of the value it would have achieved in the absence of δx_0. Now imagine a second variable $x(\mathbf{r}_2, t)$ at point $\mathbf{r}_2$ separated from $\mathbf{r}_1$ by distance R. The locality of the rule implies that information about the value of $x(\mathbf{r}_1, t)$ cannot be transmitted to other parts of the system faster than with some maximum velocity, c say. (In a cellular automaton with a nearest

[5] The coupled map lattices considered in ref. 8 and by many other authors (see, e.g., Kaneko[10] and Keeler and Farmer[11] are particularly convenient examples of such many-body dynamical systems.

neighbor rule, e.g., c would be one lattice spacing per time step.) Until time $t \sim R/c$, therefore, the variable $x(\mathbf{r}_2, t)$ evolves in ignorance of the fact that $x(\mathbf{r}_1, t)$ has experienced a perturbation. Thus, if $R/c > t^*$, then $x(\mathbf{r}_1, t)$ is already dephased before $x(\mathbf{r}_2, t)$ finds out about the perturbation. Hence, $x(\mathbf{r}_2, t)$ has no knowledge of the value of $x(\mathbf{r}_1, t)$; the perturbation has decorrelated the two variables. Similarly, of course, it decorrelates $x(\mathbf{r}_1, t)$ from *any* variable distant from $\mathbf{r}_1$ by more than the "coherence length" ξ defined by

$$\xi \equiv (c/\lambda) \ln(\Delta x/\sigma) \tag{2}$$

where $\sigma \sim \delta x_0$ measures the strength of the perturbation.

In realistic many-body dynamical systems, noise of any type (e.g., thermal) provides a steady source of such decorrelating perturbations. It is important to note, however, that even in noiseless systems, any randomness in initial conditions will likewise provide a perfectly adequate decorrelating effect, i.e., a nonzero σ. We conclude, therefore, that under generic (i.e., random) initial conditions, not even deterministic dynamical systems can, if they are chaotic, sustain spatial correlations over distances longer than ξ of Eq. (2).

The finiteness of spatial coherence lengths in locally chaotic systems implies the nonexistence of "collective chaos"—temporal chaos in the amplitudes of spatially extended (e.g., Fourier) modes of the system. The point is that such amplitudes involve spatial averages over the entire system (i.e., incoherent averages over many essentially uncorrelated regions of linear size ξ); such averages cannot produce a chaotically varying result. Naively one would expect them to give rise only to stationary, or time-independent, answers. This is often the case. We pointed out in Ref. 8, however, that if the individual variables move periodically between bands of allowed values, the precise value achieved within each band being a chaotic function of time,[12,13] then the incoherent averages can produce a periodic result which reflects the regular periodic motion between bands. Thus, the behavior of the locally chaotic system on the macroscopic level is either stationary or periodic, not chaotic.

It is important to apply this result carefully to finite-size systems. Only in the thermodynamic limit is the averaging away of chaotic behavior in Fourier amplitudes complete; in any system with finite linear size L these amplitudes (appropriately normalized) will show chaotic variations roughly of $O[(L/\xi)^{-d/2}]$ in d dimensions. Obviously when ξ of Eq. (2) is comparable to or greater than L, the system will behave chaotically even on its largest length scale. (In ref. 8 we showed an explicit numerical example of this for coupled map lattices of varying size.) The regime $\xi \geqslant L$ is readily achieved in practice: Eq. (2) shows that ξ grows without bound as λ decreases. Values of λ can be very small, particularly just above the onset of chaos, where many measurements are performed. (In the period-doubling route to chaos, e.g., λ goes continuously through 0 at the onset of chaos,[14,15] and so can be arbitrarily small.) If the velocity c with which information propagates is large, moreover, then even moderately large λ's can give rise to large ξ's, and hence to apparent collective chaos in finite samples. Fluid mechanical systems present an additional complication, in that the typical time scales (and hence typical values of λ) can vary inversely with L.[16],6 Hence, correlation lengths presumably obey a scaling relation something like $\xi \sim Lf(R)$ for some function f, R being a dimensionless quantity such as a Reynolds or Rayleigh number,[16] depending on the system at hand. In this case, one's ability to probe the regime $L \gg \xi$, where chaos on the largest length scales averages away, depends on $f(R)$ becoming vanishingly small as R increases. If this is the case, then, since R typically increases with L, increasing L with the other parameters fixed suffices to make L/ξ large.

3. ATTRACTOR DIMENSIONS

The finiteness of ξ also implies that any system of linear size $L > \xi$ in d space dimensions contains roughly $(L/\xi)^d$ effectively independent degrees of freedom. It is precisely this quantity—the number of independent degrees of freedom contributing to the chaotic evolution—that each of the several different definitions of attractor dimension D attempts to quantify (see, e.g., ref. 17, and references therein). We conclude that D must blow up roughly like $(L/\xi)^d$ as L becomes large.[7]

It is easy to understand the phenomenon of "low-dimensional chaos," i.e., very small attractor dimensions measured in chaotic many-body systems,[18] in this context. Low-dimensional chaos occurs when ξ is comparable to the system size L, a situation which we have just seen is readily realizable in practice, either because λ is small or c large. Except in situations where ξ increases linearly with L, therefore, low-dimensional chaos is a consequence of insufficiently large sample size: Increasing L beyond ξ will result in D increasing like $(L/\xi)^d$, as in the experiments cited in footnote 7.

[6] We are grateful to P. C. Hohenberg for a helpful comment on this point.

[7] Recent numerical experiments of Y. He and C. Jayaprakash (unpublished) on two-dimensional coupled lattice maps in the chaotic regime show attractor dimensions growing like L^2 with increasing system size L, consistent with the general expression $D \sim (L/\xi)^d$ in d dimensions.

4. PERIODIC STATES

Let us now turn to periodic states, restricting ourselves to systems that evolve in discrete time according to local, dynamical rules with (discrete) time-translation invariance. We try to make clear why, in the presence of any noise, stabilizing collective periodicity is difficult, and to state the conditions under which it is possible, referring the reader to ref. 9 for details and specific examples.

The difficulty inherent in the stabilization of periodic states is easily understood by analogy to the nucleation and growth of the unique stable phase in the classical Ising model in a small magnetic field at low but nonzero temperature. Let us therefore briefly review the familiar phenomenon of nucleation and growth (see, e.g., ref. 19, and references therein): Suppose that the field points in the up direction; thus, we expect the single stable phase of the model to have positive magnetization. Suppose, however, that we prepare the system in an initial condition wherein all spins point downward, and let the system evolve according to some stochastic dynamical rule that satisfies detailed balance for the Ising Hamiltonian.[20] The system will, due to the thermal noise, nucleate droplets of the favored up spins. If the temperature T is low, then initially the droplets will be small and rather far apart. It is well known that the evolution of an isolated droplet with radius R (Fig. 1) is described in any dimension by the phenomenological equation[21,22]

$$dR/dt = -\sigma/R + h \tag{3}$$

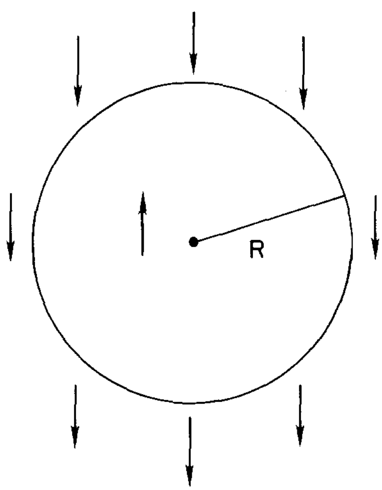

Fig. 1. Droplet with radius R of Ising up spins immersed in a sea of down spins at low temperature.

where σ is roughly the surface tension, and h is proportional to the magnetic field. The first term on the right side, proportional to the curvature, $1/R$, represents the tendency of the droplet to shrink in response to the surface tension, thereby reducing the area of the boundary (domain wall) between the up and down phases. The second term, h, is the average velocity with which an infinite, flat domain wall between the two phases translates. [To see this, let R grow very large in Eq. (3), whereupon the left side represents the speed with which an essentially flat wall moves, and the right side is simply h.] In this example, h is positive, since we have assumed that the field points up, favoring growth of the droplet. It is well known (or readily seen by inspection) that Eq. (3) describes the respective growth or shrinking of droplets whose radius R is larger or smaller than a critical size $R_c \equiv \sigma/h$. Eventually [the time required being of order $\exp(\sigma R_c^{d-1}/T)$ in d dimensions], the system will nucleate a finite density of spin-up drops with radii greater than R_c. Equation (3) shows that the drops will then expand linearly in time, thereby replacing the spin-down initial state by the unique, stable, equilibrium state with positive magnetization. So long as there is any noise, the system attains this stable state in the long-time limit regardless of the initial conditions.

The crucial point is that the field—however small—favors the spin-up state, thus breaking the up–down symmetry of the zero-field problem. The inequivalence of the spin-up and spin-down states is manifest in the finite velocity [h in Eq. (3)] with which a flat interface between the two states translates. On the coexistence curve, $h = 0$, of the Ising model at low temperature, the up and down spin states become equivalent. In consequence, flat interfaces do not translate at all [see Eq. (3) with $h = 0$ and $R \sim \infty$]. Finite droplets of any initial radius R_0 and either sign shrink to zero like $R^2 = R_0^2 - 2\sigma t$ under the action of the surface tension. Thus, the critical radius R_c is infinite, consistent with the existence of two equivalent, stable, broken-symmetry states of the system with equal and opposite magnetizations. Starting from an initial condition consisting predominantly of up (down) spins, one achieves in the long-time limit the stable equilibrium state with positive (negative) magnetization. We emphasize that for systems described, like the Ising model, by a Hamiltonian, this is possible only on the coexistence curve—i.e., on a set of zero measure of the Ising phase diagram in the h–T plane. The generic situation ($h \neq 0$) is characterized by nucleation and growth of a unique stable state.[22]

Let us now return to the problem of stabilizing periodic states, first restricting ourselves to spatially isotropic systems. Imagine that we have suceeded in constructing a (discrete-time) dynamical rule which has broken its time-translation invariance by producing, say, a stable 3-cycle. That is, imagine that some Fourier amplitude, say the amplitude of the $k = 0$ mode

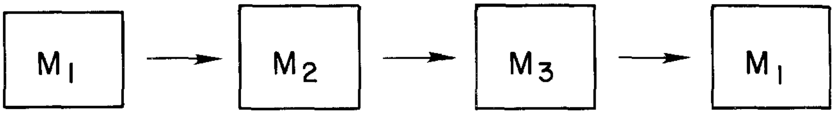

Fig. 2. A periodic 3-cycle state at four successive time steps. At the first, second, and third time steps, most of the system variables assume the values M_1, M_2, and M_3, respectively. At the fourth step, M_1 occurs again.

(the spatial average M of the system's dynamical variables), assumes three distinct values, M_1, M_2, and M_3, say, in regular periodic fashion at consecutive time steps (Fig. 2). The rule is assumed noisy, so that no single variable will assume this precise, repeating sequence of values; only the spatial average (or, equivalently, the noise-averaged value of any given variable) in the thermodynamic limit is truly periodic.[8]

Suppose that we start the system off in an M_1 initial state, and monitor its evolution at every third time step thereafter. If the evolution is genuinely periodic, then the system viewed at three-step intervals should look stationary, i.e., the M_1 state should persist indefinitely. In the presence of any (Gaussian or other unbounded[9]) noise, the system will of course nucleate droplets of M_2 in the infinite sea of M_1. The stability of the periodic time dependence is determined by considering the dynamics of such droplets, whose evolution (monitored at every third step), proceeds[(22)] according to the same phenomenological equation (3) that describes drops in the Ising model. Again the term on the right side of (3) proportional to the droplet's curvature, $1/R$, expresses the tendency of the droplet to shrink and so reduce the length of the domain wall between the phases; σ represents the analogue, for nonequilibrium systems, of surface tension. The second term, h, is again the average velocity with which an infinite, flat domain wall between the two phases translates in the course of three steps. As in the Ising model, this term is a measure of the inequivalence of the states M_1 and M_2. The main point is that, again as in the Ising model, the generic situation is the one where there is no symmetry guaranteeing the equivalence of M_1 and M_2, i.e., where h is nonzero in Eq. (3). While it is possible to construct rules with a symmetry that ensures that walls do not translate on average (an example is given in ref. 9), the typical case, and certainly the overwhelmingly probable one encountered in practice, is the one where the states are inequivalent. (It is easy to check, e.g., for the

[8] Refs. 8 and 9 contain more detailed discussions of periodic states in the context of specific dynamical systems, namely coupled map lattices and cellular automata, respectively.

[9] Noise of bounded amplitude may be insufficient to nucleate droplets, but such noise is rather unphysical. In equilibrium statistical mechanics, e.g., finite systems with bounded noise can exhibit broken symmetries, an impossibility with the more realistic unbounded noise.

periodic states studied numerically in the coupled map lattices of ref. 8, that no such symmetry is present.[10])

Let us assume for the moment that h is positive, so that M_2 is favored over M_1. Then, just as in the Ising model, the system eventually nucleates droplets larger than the critical size $R_c \equiv \sigma/h$, which expand to replace the state M_1 by M_2. After sufficiently long time, therefore, the state M_1 fails to occur at every third time step. In other words, the assumed temporally periodic state of the system is not stable, but *metastable*. One might think that the M_2 state which supplants M_1 simply persists indefinitely (still viewed at each third time step), thereby implying that the system is still in a stable 3-cycle, but shifted in phase from the initial condition. This is not the case: Since in one time step M_1 turns into M_2, and M_2 turns into M_3, the instability of M_1 with respect to droplets of M_2 implies the instability of M_2 with respect to droplets of M_3 (and also the instability of M_3 to droplets of M_1). Thus, starting as we did from a state of pure M_1, one finds that large droplets of M_2 begin to grow and supplant M_1 after many 3-step cycles; but M_2 itself then starts being supplanted by M_3, etc. After many cycles, therefore, one expects to find a mixture of equal parts of M_1, M_2, and M_3 in the system at any given time. In other words, the overall spatial average M of the variables in the system ceases to vary with time, and the behavior of the system is *stationary*, not periodic.

Note that the instability of the periodic state does not depend on our assumption of positive h in Eq. (3) for the evolution of drops of M_2 in a sea of M_1. If h is in fact negative, so that M_1 is favored over M_2, then droplets of M_1 immersed in a sea of M_2 evolve according to Eq. (3) with h positive; we can simply repeat the arguments above for droplets of this type, with the same result, namely the destabilization of the assumed periodic state. Thus, the nucleation and growth mechanism of Eq. (3) destroys collective periodicity under all generic conditions. It is clear from (3), however, that a periodic state can readily be made *metastable*, its lifetime roughly given by the time (a function of the noise, h, and σ) required for the nucleation of droplets of critical size. This time grows exponentially with R_c. If the velocity h is small, so that R_c is large, the lifetime of the periodic state can be very long indeed.[11]

[10] Models commonly taken as prototypes for studying the onset of periodic behavior, such as the Brusselator,[23] also lack this symmetry, as do typical experimental systems that exhibit periodic states.

[11] We now believe, e.g., that the periodic states observed numerically in the coupled maps of ref. 8 (on samples of up to 200×200, and hundreds of thousands of updatings per spin), were actually metastable rather than stable. By measuring the rate [h in Eq. (3)] at which flat domain walls move in the 4-cycle regime of that reference, we estimate that the critical droplet size for destruction of the periodicity is of $O(3000)$, far beyond our numerical grasp.

Only by imposing a symmetry that guarantees the equivalence of M_1, M_2, and M_3, or by making special parameter choices that guarantee that flat domain walls move with zero velocity[9] ($h=0$), can one stabilize periodic states in (3). This is analogous to positioning oneself on the coexistence curve of the Ising model, where $R_c = \infty$, and droplets of any initial radius R_0 shrink like $R^2 = R_0^2 - 2\sigma t$.

Though we have discussed only 3-cycles, it seems clear that similar arguments prohibit the existence of stable cycles of any arbitrary length in isotropic, generic, discrete-time systems with short-range interactions. The single exception is the 2-cycle, where the fact that the two states, M_1 and M_2 say, exchange identities at each time step implies that a flat domain wall separating them cannot translate with nonzero velocity.[8] Thus, unlike higher cycles, the 2-cycle has an effective symmetry that ensures the vanishing of h in Eq. (3), and so allows the stabilization of the period-2 state, even under generic conditions.

In writing Eq. (3) and so arguing against stable cycles with periods longer than 2, we implicitly exploited the assumed spatial isotropy of the system. We now show how systems (e.g., on a lattice) without isotropy *can* exhibit stable states with arbitrary periodicity.[9] Imagine that the 3-cycle system described above is defined on, e.g., a square lattice in two dimensions. While it remains true that flat domain walls separating any two of the phases M_1, M_2, and M_3 must translate under generic conditions, the

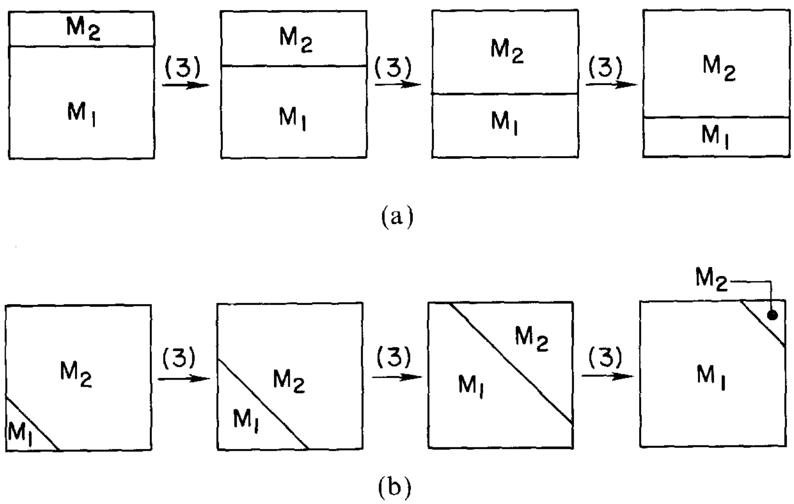

Fig. 3. Translation, in each 3-step cycle, of flat domain walls separating a domain with variable value M_1 from one with value M_2. Domains oriented (a) parallel or (b) at 45° to the lattice axes translate to favor (a) M_2 over M_1 or (b) M_1 over M_2.

sense in which they translate can depend on their orientation with respect to the lattice axes. It is easy, e.g., to construct rules (a specific example is given in ref. 9) wherein walls between M_1 and M_2 oriented parallel (at 45° to) the axes translate to favor M_2 over M_1 (M_1 over M_2), as in Fig. 3. For finite droplets, this has the effect shown in Fig. 4: A droplet of M_1 in a sea of M_2 grows fastest along those parts of its boundary oriented at 45° to the lattice axes. Thus, it distorts into a rectangular shape whose boundaries are oriented along the lattice directions. Since domain walls with lattice-axis orientation move to favor M_2, the droplet then shrinks and vanishes! (For large droplets this situation can be described phenomenologically by the equation $dR/dt = -h$, with h positive. Thus droplets shrink *linearly* with time.) Similarly, a droplet of M_2 in a sea of M_1 distorts so that its boundaries orient at 45° to the axes, whereupon it also shrinks and disappears. Thus, systems with appropriate rules can take advantage of spatial anisotropy to ensure that droplets created by thermal fluctuations do not grow and destroy the temporal periodicity. Such systems can therefore exhibit stable 3-cycles, and by obvious generalization, higher cycles as well.

Experimentally realizable dynamical (e.g., fluid mechanical) systems typically occur in the continuum rather than on a lattice. However, the stabilization mechanism described above requires only that there be some

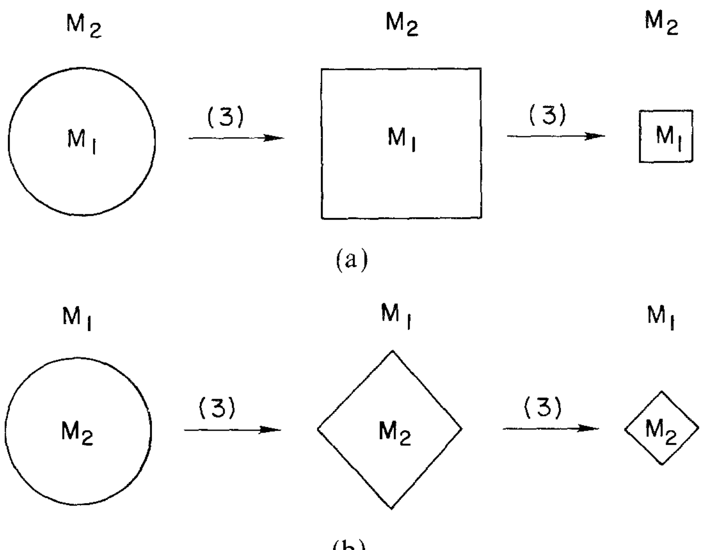

Fig. 4. (a) Droplet of M_1 in a sea of M_2 seen at 3-step intervals. The droplet distorts into a square oriented with lattice axes and then shrinks. (b) Droplet of M_2 in a sea of M_1 seen at 3-step intervals. The droplet distorts into a square oriented at 45° to the lattice axes and then shrinks.

special direction in the problem, so that domain walls oriented in different directions need not translate in the same sense. Such anisotropy is a ubiquitous feature of (even fluid mechanical) systems that have periodic states. In Rayleigh–Benard systems, e.g., the occurrence of periodic states occurs only at Rayleigh numbers above the convective instability where the system first develops rolls.[2] The axes of the rolls define a preferred direction which the system chooses spontaneously, thereby breaking its spatial isotropy. The system could, in principle, then make use of anisotropy as described above to produce stable periodic states. (We emphasize, however, that the details of the stabilization of periodic states have not yet been fully worked out for continuous-time systems, though we suspect the main results are similar to the discrete-time situation studied here.)

It is important to note that only certain rules can successfully exploit spatial anisotropy to stabilize periodic phases. (The general rquirements are made clear in ref. 9.) For example, we now believe that the periodic states we observed numerically in the coupled map lattices of ref. 8 are not genuinely stable, but rather metastable with a lifetime orders of magnitude larger than the longest computer runs currently feasible (see footnote 11). Similarly, metastable periodic states may readily be mistaken for stable ones in real experiments. It is interesting to ask, e.g., whether the periodic states observed in Rayleigh–Benard and other systems use the above mechanism to achieve true stability, or whether they are simply metastable, with lifetimes long compared to typical measuring times. We have more to say about distinguishing between these two possibilities elsewhere.

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